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Schanuel's theorem for heights defined via extension fields

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Abstract. Let k be a number field, let θ be a nonzero algebraic number, and let $H(\cdot)$ be the Weil height on the algebraic numbers. In response to a question by T. Loher and D. W. Masser, we prove an asymptotic formula for the number of $\alpha \in k$ with $H(\alpha\theta) \leq X$, and we analyze the leading constant in our asymptotic formula. In particular, we prove a sharp upper bound in terms of the classical Schanuel constant.

We also prove an asymptotic counting result for a new class of height functions defined via extension fields of k with a fairly explicit error term. This provides a conceptual framework for Loher and Masser's problem and generalizations thereof.

Finally, we establish asymptotic counting results for varying θ, namely, for the number of $\sqrt{p}\alpha$ of bounded height, where $\alpha \in k$ and p is any rational prime inert in k.

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